Aristotle on the Objects of Natural and Mathematical Sciences

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Abstract

In a series of recent papers, Emily Katz has argued that on Aristotle's view mathematical sciences are in an important respect no different from most natural sciences: They study sensible substances, but not *qua* sensible. In this paper, I argue that this is only half the story. Mathematical sciences are distinctive for Aristotle in that they study things 'from', 'through' or 'in' abstraction, whereas natural sciences study things 'like the snub'.

What this means, I argue, is that natural sciences must study properties as they occur in the subjects from which they are originally abstracted, even where they reify these properties and treat them as subjects. The objects of mathematical sciences, on the other hand, can be studied as if they did not really occur in an underlying subject. This is because none of the properties of mathematical objects depend on their being in reality features of the subjects from which they are abstracted, such as bodies and inscriptions. Mathematical sciences are in this way able to study what are in reality non-substances as if they were substances.

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I. Introduction

The ontology of the objects of mathematical sciences presents a special problem for Aristotle. On the one hand, terms for shapes and numbers function as substantives. Accordingly, Aristotle often uses them as

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examples of subjects in the *Posterior Analytics*, suggesting that they are substances rather than attributes.[1] Aristotle is however acutely aware that mathematical objects differ in a fundamental way from his other paradigms of substances. They are neither like animals, conceived of as hylemorphic compounds, nor like divine substances, thought of as pure form without matter, and this makes the question whether mathematical objects are substances a live one.

Aristotle raises this question in *Metaphysics* B.2, attributing the view that mathematical objects are substances to those philosophers who 'make mathematicals intermediate between forms and perceptibles', a Platonic position he wishes to avoid (*Met.* 995b.16–18, my translation).[2] The issue is treated as resolved in *Metaphysics* Λ.8, where, in passing, Aristotle asserts that the mathematical science of astronomy studies eternal perceptible substances, whereas the purer branches of mathematics, arithmetic and geometry, 'do not study any substance' (οὐδεμιᾶς οὐσίας, *Met.* 1073b7, Judson trans.). In other contexts, however, he indicates there is more to be said. In his tripartite division of theoretical sciences into physical, mathematical and theological in *Metaphysics* E.1, the issue of whether mathematical objects are separate, a defining feature of substance, is flagged as 'presently unclear' (νῦν ἄδηλον, *Met.* 1026a8–9; cf. 1059b.1–12) and we find Aristotle hedging in the same way in Z.2 and H.1, deferring the investigation of 'mathematical objects and ideas' (*Met.* 1042a22; cf. 1028b19–31) to a later study.

It is widely agreed that Aristotle undertakes this joint investigation of ideas and mathematical objects in *Metaphysics* M and that his answer to the substance question there is negative.[3] Mathematical objects – at least if we discount those of 'mixed' mathematical sciences like astronomy, as I will from here on in – are not, all things considered, substances for Aristotle. There is less agreement regarding his positive proposal, and how mathematical objects are ultimately related to sensible substances for him.

Emily Katz has argued that for Aristotle mathematical objects exist as properties of sensible substances (Katz 2019), or, in more recent expositions of her view, as accidental compounds or 'kooky objects' that involve sensible substances (Katz 2022, 2023).[4] A kooky object is a thing like the sitting Socrates. This is an object which, on Aristotle's view, inheres in Socrates and exists only when Socrates is sitting. Likewise, mathematical spheres are things that inhere in sensible bodies as their limiting boundaries and divisions and exist only when those sensible objects are bounded and/or divided in the relevant way. Geometrical spheres, for instance, inhere in sensible objects such as ball bearings and exist just when the relevant lump of metal bounds a spherical volume.[5] Similarly the arithmetician deals with numbers that exist like 'the 1,000 leaved thing' (Katz 2023: 137) – an object that inheres in the foliage and exists exactly when it consists of 1,000 leaves.

As Katz stresses, to this extent mathematical sciences are like a variety of others that treat their subjects only in so far as they have certain properties. The science of human medicine treats human health. Health, on Aristotle's view, is not a separately existing entity or form; it is a feature of living things, and human health is specifically a feature of humans. Hence human medicine is, on Aristotle's view, a study of human beings. Yet that does not mean that it treats any and all features of human beings, or that it studies human beings as such. Rather, it treats human beings in so far as they are healthy (*Met*. 1077b34–1078a2). Likewise, for Aristotle mathematical sciences study perceptible things, but only in so far as they have certain mathematizable features (1078a2–5).

In at least this respect, then, geometry and arithmetic are unremarkable sciences for Aristotle. They study sensible things, but not *qua* sensible. I think, however, that the analogy between mathematics and other sciences tells only half of the story, and in the absence of further context it may mislead us into thinking that mathematical sciences are for Aristotle more similar to other sciences than they actually are.

In particular, I maintain, the attitude of the mathematician differs from the natural scientist not only in what she abstracts but in how she views the product of that abstraction in relation to its original substratum. Mathematics permits a complete reification of its abstractions and thus the mathematician may proceed as if her abstractions were substances. While a certain limited reification of non-substances is also legitimate and indeed necessary in natural sciences, Aristotle takes abstraction in natural sciences to have limitations and dangers requiring the scientist to attend, in certain contexts, to the substratum from which an object is originally abstracted. It is characteristic of mathematics that these limitations do not apply, which is why Aristotle extends the moniker of 'things from abstraction' to mathematical objects but not objects of natural sciences. Hence even if Aristotle does set up a 'tight analogy'[6] between the objects of mathematics and other sciences *vis-à-vis* their status as things that ultimately inhere in sensible objects, there is an equally important disanalogy: Mathematicians are able to bracket the fact that the properties under investigation inhere in a subject in a way that other sciences cannot.

The view that mathematical objects are 'quasi-substances' for Aristotle is not new. Halper (1989, 252) talks of the 'quasi-substantial character' of mathematical substances, although he explicitly takes this to be a feature that mathematicals share with other non-substantial subjects like houses, cities and plays. Meanwhile, Corkum (2012) develops the view that Aristotle is a fictionalist about the ontological status of mathematical objects, but not about mathematical truth. I wish to go beyond these accounts by explaining what, for Aristotle, this special status of mathematical objects as quasi-substantial consists in, and how a view of mathematical objects as distinctively abstract is nevertheless compatible with Katz's view of mathematical objects as inhering in sensibles. Indeed, I argue, it is only by seeing the way that mathematical objects are alike *and* unlike the objects of natural sciences that we gain a proper view of Aristotle's big picture when it comes to mathematics.

I proceed as follows. First (section II), I examine how Aristotle assimilates the objects of mathematical sciences to those of natural sciences in *Metaphysics* M.3. In section III, I argue that while this yields one sense in which mathematical sciences involve abstraction, this cannot be what Aristotle means when he refers to mathematical objects as things 'from abstraction (ἐξ ἀφαιρέσεως)' or 'in abstraction (ἐν ἀφαιρέσει)'. Finally (section IV), I explain how Aristotle views the limits of abstraction performed in physical sciences, so as to show, *via negativa*, in what way Aristotle takes abstraction to be safe in mathematics while risky elsewhere. This will reveal the distinctive way in which mathematical objects are abstract for Aristotle.

II. Aristotle on the Objects of Sciences in *Met.* M.3

A unifying concern of *Met.* M is the ontological status of mathematical objects (τὰ μαθηματικά). *Met.* M.1 raises two structuring questions: Do mathematical objects exist (1076a32–33)? And if so, do they exist in sensibles (1076a33–34)? Aristotle approaches these questions in M.3 from the vantage point of the mathematical sciences, treating the objects of mathematics as the things sciences are 'of', as when one calls arithmetic a science of numbers (τῶν ἀριθμῶν, 1076b36, cf. 1077b35) or geometry a science of shapes. He argues that such objects exist and, furthermore, that they exist in sensible things.

In order to understand Aristotle's arguments in M.3, it is important to see that he assumes an audience that takes the objects of natural sciences like medicine and zoology to exist unproblematically as features of natural things. His approach is to minimize the extent to which there is a special problem regarding the existence and status of the objects of mathematical sciences. If successful, the sceptic will be forced to acknowledge the existence of mathematical objects and concede that they too exist in sensible things on pain of denying the same about zoology, medicine, etc.

A key move in his argument is to lay down a principle regarding the identity of the objects of a science:

> It is true to say of other branches of knowledge, without qualification, that they are of this or that – not of what is incidental (e.g. not of the white, even if the branch of knowledge is of the healthy, and the healthy is white) but what each branch of knowledge is of, the healthy if <it studies its subject> as healthy, man <if it studies it> as man. (*Met.* 1077b34–1078a2)[7]

According to this passage, the objects of sciences are determined by that *qua* which they study the things they study. Specifically:

> A science S is of the X if S studies things *qua* X (*qua* principle)

Determining what a science is of, then, turns out not to be as simple as determining the extension of things it studies (where this means, roughly, those things which are examined in order to ascertain the results of that science and to which these results apply).[8] Medicine studies what is healthy, and certain healthy things are white, but that doesn't make medicine a science of the white – neither in the sense that it is a science of whiteness in general nor in the sense that it is a science of some white thing or things. Nor is medicine a science of the animal, even though all healthy things are animals (and even essentially so). Rather, the object of medicine is determined by the aspect under which these things are studied, namely health.

The import of this claim is best grasped by seeing the alternatives which it is designed to avoid.[9] On the one hand, Aristotle holds that sciences cannot be of sensible particulars. One reason is a worry about the precision of mathematical sciences: Mathematical sciences speak of breadthless planes that touch a sphere at one point, but 'no perceptible thing is straight or curved in this way' (*Met.* 998a1–2, trans. Pettigrew). Another is that he takes sciences to be relatives, and thus to depend for their existence on the existence of what they are of (*Cat.* 6b5). Sciences, however, can be taught, learned and retained regardless of whether any particulars to which they apply are presently in existence,[10] hence their objects – what they are 'of' – cannot be perishable particulars (*Met.*

1039b31–40a7), and this holds in particular in the case of mathematical science (*Met.* 997b32–4, 1059b10–12).

Both these considerations require distinguishing sensible substances and mathematical objects. Indeed, the latter consideration might seem to imply that no sciences study sensible things. Aristotle however rejects this position, which he takes to be Platonic (*Met.* 1078b12–17). In particular, he takes mathematics to be one of the sciences that studies the sensible (1078a2–8), seemingly moved by the fact that mathematics is, as Jonathan Lear puts it, 'so richly applicable to the physical world' (Lear 1982: 189). He is able to render these views consistent by rejecting the assumption that the objects of a science are to be identified with the extension of things it studies.[11] Mathematics *does* study sensible substances, but since it is not therefore a science *of* these sensible substances, mathematical knowledge is never at the mercy of whether certain perceptible particulars continue to exist. While sensible triangles and circles may suffer various forms of imprecision, the objects of mathematics are themselves precise.

Of course, one might ask whether Aristotle has simply moved the problem around, since now he needs to explain the status of these objects that sciences are 'of' without making them, in turn, into further substances. He also needs to explain how these objects can be precise given the imprecision of sensible geometrical objects.[12] Notice, however, that the force of his argument in M.3 for the existence of mathematical objects doesn't depend on meeting these challenges. So long as his opponent is willing to accept, first, that the objects of natural sciences exist because natural scientists study specimens *qua* some determinate feature, and, second, that the same holds true of mathematics, he will have succeeded in getting his opponent to admit that the objects of mathematical sciences exist. He doesn't need to give a positive account of the ontology of either mathematical objects or the objects of natural sciences in order to secure this admission. The more difficult claim to establish is that mathematical objects, like objects of natural sciences, exist in sensible things.

As is widely appreciated, Aristotle has a more fine-grained ontology than most philosophers today.[13] Given a body that is in fact spherical, he will distinguish between the body, on the one hand, and the spherical thing on the other. The differences between the two objects show up in their respective modal profiles: While the body can exist even if deformed into another shape, the spherical thing cannot. Rather than taking the spherical body to be simply identical with the body and distinguishing *de re* and *de dicto* statements of identity, Aristotle treats the body and the spherical thing as two beings, albeit with certain relations of sameness obtaining between them (*Met.* 1017b30, 1024b30–31). Objects like 'the spherical things' have become known in the literature as 'kooky objects' since Matthews (1982).

Katz (2022) argues that Aristotle exploits this ontology in order to vindicate the claim that mathematical objects exist in sensible things. In claiming that geometry studies sensible things, but not *qua* sensible (1077b22, 32–33), Katz takes Aristotle to be claiming that geometers study kooky objects like the spherical thing. This object is not simply identical to the object of natural science, the body; but nor is it something 'separate' (χωριστόν) from it or 'over and above' (παρά) it (cf. *Met.* 1026a15, 1059a8–9, 1078a7–8; *Post. An.* 77a5). On Aristotle's view, not only properties like being spherical but also kooky objects can inhere in substances: The spherical thing is 'in' a body, not as a spatial part but as an accident. It inheres in the technical sense of 'being in' defined in *Categories* 2.[14]

This fine-grained ontology thus allows Aristotle to thread the needle between Platonism and the identification of mathematical objects with sensibles. A 'kooky object' like the spherical thing is neither identical with the sensible body nor a distinct substance. Likewise, the objects of arithmetic are particular, sensible multiplicities, though not considered *qua* sensible. These, too, are not simply identical to sensible objects like the foliage, but nor are they separate from them: The 1,000 leaves inheres in the foliage as an accident.[15]

This solution, however, raises a further puzzle that is perhaps more pressing than Katz recognizes. If Aristotle's motivation for resisting the identification of sensible particulars with mathematical objects stems in part from a concern with their perishability, hasn't he only made matters worse by making mathematical objects into accidental compounds of sensible substances and contingent properties? After all, as a kooky object the spherical thing *depends* for its existence on its bearer, the sensible body.[16] It consequently has all of the ontological precarity of the body and more. It perishes not only when the body does, but when the body ceases to be spherical.

One aspect of Aristotle's solution is to distinguish between the way that mathematical knowledge is of the universal and the way it is of the particular, a task he undertakes in *Met.* M.10.[17] Even if a particular spherical thing can perish, this does not mean that the universal sphere, and truths about it, can too. This is the solution Katz favours,[18] and as she notes, this manoeuvre may serve to bolster the stability of any science for Aristotle by insulating it from the contingency of its individual objects. Yet one hopes that this is not all Aristotle has to say in response to this motivating concern of his philosophy of mathematics. Whereas a zoologist will not be surprised that the explanation of digestion no longer applies to the animal after dissection, a mathematician does not expect the mathematical triangle which was the subject of a construction suddenly to have a different angle sum, or none at all, when the construction drawing is destroyed. A further problem is that mathematical proofs frequently make reference to multiple mathematical objects of the same kind, all of which need to have the requisite stability in order to serve as objects of mathematical proof, and so Aristotle cannot simply say that mathematical statements are about *kinds* of mathematical objects.[19]

On Katz's interpretation, therefore, Aristotle lacks an adequate solution to this foundational problem. Charity might lead us to question, then, whether mathematical objects are accidental compounds for Aristotle after all. I believe that a more adequate solution to this problem can however be drawn from Aristotle's texts without rejecting this crucial commitment of Katz's view. First, we need not suppose that for Aristotle all geometrical properties exist as *actual* properties of bodies. An alternative is to view mathematical objects as sensible particulars compounded with various actual-or-potential properties, so that a geometrical sphere, for instance, is an accidental compound of body and actual-or-potential sphericity. In this case, the relevant compound can continue to exist even after deformed since its potential sphericity is not removed by the deformation.[20]

III. Abstraction in *Post. An.* I.5 and Other Texts

We have seen that Aristotle wishes to avoid Platonism by maintaining that mathematical objects inhere in, but are not to be identified with, sensible things. Whatever Aristotle means by mathematical objects being abstractions, then, it cannot be that they are non-sensible objects if we are to take Aristotle to have a consistent view.[21]

As Katz and others have noted,[22] Aristotle gives an account of what it means to study X *qua* Y in *Posterior Analytics* I.5, and this account makes crucial use of the terminology of abstraction at 74a37–8. He discusses a range of cases where we can be misled into thinking that X holds of Y *qua* Z when it does not, such as if we manage to prove the 2R theorem[23] of an isosceles triangle and conclude, erroneously, that it holds of shapes in so far as they are right-angled triangles rather than simply in so far as they are triangles (*Post. An.* 74a16–17; cf. 25–32).[24]

In general, Aristotle holds, a science studies something *qua* F if it demonstrates that the properties of whatever it studies hold of it *qua* F. This requires him to give an account of when X holds of Y *qua* Z, which he does by saying that it holds if X holds of Y and Z is the 'first item after the removal of which (ἀφαιρουμένων)' X no longer holds of Y (*Post. An.* 74a37–8). His example is that 2R holds of bronze isosceles triangle *qua* triangle. 2R will hold of a bronze isosceles triangle even if we assume that it is not bronze, and it will still hold even if it is assumed not to be an isosceles, but it will no longer hold if it is assumed not to be triangular. 'Triangle' is thus the first predicate after the removal of which 2R no longer holds of this object and, consequently, 2R holds of it *qua* triangle.

While Aristotle's procedure in the example assumes a strict ordering of predicates from least to most general, it is clear that the criterion that he is laying down doesn't actually require this. Writing *S–F* to mean '*S* after the removal of *F*', Aristotle's point is that *P* holds of *S qua F* just if (a) *P* holds of *S*, (b) *P* does not hold of *S–F*, and (c) for every *F′* less general than *F*, *P* holds of *S–F′*.

Notably, Aristotle uses the verb 'abstract' or 'subtract' (ἀφαιρέω) to describe the removal of predicates in this context (74a37–8). This yields a sense in which natural and mathematical sciences alike involve removal or abstraction: They study objects *qua* some feature of them (*Met.* M.3), and studying-*qua-F* is to be understood in terms of what holds of a subject *qua*-F, which in turn is to be understood in terms of what holds of S after the removal or abstraction of predicates (*Post. An.* I.5). Plausibly, this is one part of what Aristotle means elsewhere when he describes the objects of mathematics as abstracted. Even though what the mathematician studies are physical bodies (*Phys.* 193b23–25), the mathematician separates these from perceptible matter (*Met.* 1025b34) and from motion (*Phys.* 193b33–34, *Met.* 1077b28), and one thing this means is that she does not study things *qua* changeable or *qua* sensibly-enmattered in the sense just described (cf. *Met.* 1077b23–30). These are after all not the features that a mathematician appeals to in order to construct mathematical demonstrations and hence they do not define what mathematical sciences are of according to the *qua* principle.[25] Rather, mathematicians are interested in sensible substances *qua* quantitative and continuous (*Met.* 1061a34–5).[26]

This however cannot be all Aristotle means by talking about abstraction in connection with mathematics, for two reasons. First, Aristotle frequently *contrasts* mathematics as a science of abstractions with natural science. Thus at *Part. An.* I.1, Aristotle claims that no natural science is 'of things from abstraction' (τῶν ἐξ ἀφαιρέσεως, 641b10–11). We cannot make sense of this claim on the assumption that studying things from abstraction simply denotes bracketing certain properties *à la Post. An.* I.5, since Aristotle's central point in *Met.* M.3 is precisely that this feature unites mathematics with other sciences, including the natural sciences. Likewise, in *NE* XI.8, Aristotle suggests (albeit tentatively) that the reason why young people can be accomplished in mathematics but not in natural science is that mathematical principles are acquired 'through abstraction' (δι' ἀφαιρέσεως, 1142a18) whereas natural sciences get their principles from experience (ἐμπειρία, 1142a19). In this connection, too, being acquired 'through abstraction' must denote a special feature of the principles of mathematics which explains why protracted experience is not needed for their cognition. Particularly clear also is *Post. An.* I.18, where Aristotle says that the objects of all sciences, '*even* those that we speak about from abstraction' (καὶ τὰ ἐξ ἀφαιρέσεως λεγόμενα, 81b3), rely in some way on induction for their cognition. Again, the 'things we speak about from abstraction' cannot be a description of the objects of all sciences, since Aristotle is singling these out explicitly as a special case in need of further argument.[27]

Other texts imply that things from abstraction are coextensive with the objects of mathematics, or at least that the objects of mathematics form a paradigm case. Thus in *De Caelo* III.1, Aristotle contrasts mathematical impossibilities, which are said of 'things from abstraction', with physical possibilities that apply to 'things from addition' (ἐκ προσθέσεως). Again, in *De Anima* I.1, III.4, III.7 and *Physics* II.2, Aristotle contrasts things 'from' or 'in' abstraction as the domain of the mathematician with those studied by the natural scientist.[28]

The second reason that mathematical abstraction must be distinctive is Aristotle's doctrine that mathematics studies objects 'as changeless and separate' (*Met.* 1026a9–10) even though they are in fact 'doubtless enmattered and not separate' (*Met.* 1026a15, trans. Kirwin). That is, Aristotle holds that the mathematician considers her objects as having the contrary of a property they actually have: They are actually enmattered but the mathematician treats them as separate.[29] This sort of abstraction has no parallel in the case of natural science. A natural scientist does not treat an animal as short-lived, for instance, just because she abstracts from its being long-lived. She simply lays aside its life span, treating it neither as long-lived nor as short-lived.

Mathematics goes further and abstracts an object that is in some respects contrary to its actual nature.[30] There is also nothing that would license such a treatment in the abstractive procedure of *Post. An.* I.5, which Jonathan Lear (1982, 168) aptly describes as a 'filter' that removes from an object its scientifically irrelevant features. While that procedure does straightforwardly license the mathematician in studying her objects not as enmattered, there is a crucial difference between studying something not as enmattered (which is what every science does except the science of matter as such) and studying something as positively separated from matter. To study a thing *qua* F is to lay aside or bracket all features of that thing irrelevant to its being F, but doing so gives no grounds for ascribing properties it does not actually possess, and *a fortiori* none for ascribing the contraries of its actual properties.

Supposing that there is more to being a thing from abstraction than simply being the outcome of the *Post. An.* I.5 procedure, then, the task is to clarify what it means to be abstract in the sense that applies to mathematical objects but not to objects of natural science, which Aristotle usually marks linguistically with a form of ἀφαίρεσις governed by a preposition ('in/from/through abstraction'). Almost all passages containing this language contrast mathematical abstractions with things like the 'snub' (σιμή). Examining this contrast is key to seeing the special sense in which mathematical objects are abstract for Aristotle.

IV. Mathematicals vs the Snub

In *Metaphysics* E.1, Aristotle classifies theoretical sciences according to a threefold scheme. Theology studies things that are separate and changeless; mathematics studies things as changeless and as separate even though they are not truly separate; and natural sciences study things like 'the snub':

> If, then, all natural things are said the way the snub is (for example, nose, eye, face, flesh, bone, and, in general, animal, and leaf, root, bark, and, in general, plant – for the account of none of these is without [reference to] change, but always includes matter), the way we must inquire into and define the essence in the case of natural things is clear.
>
> εἰ δὴ πάντα τὰ φυσικὰ ὁμοίως τῷ σιμῷ λέγονται, οἷον ῥὶς ὀφθαλμὸς πρόσωπον σὰρξ ὀστοῦν, ὅλως ζῷον, φύλλον ῥίζα φλοιός, ὅλως φυτόν (οὐθενὸς γὰρ ἄνευ κινήσεως ὁ λόγος αὐτῶν, ἀλλ' ἀεὶ ἔχει ὕλην), δῆλον πῶς δεῖ ἐν τοῖς φυσικοῖς τὸ τί ἐστι ζητεῖν καὶ ὁρίζεσθαι (*Met.* 1025b30–a4, trans. Reeve modified)

What does it mean to study something like 'the snub'? Aristotle tells us in this passage that things like the snub include matter and do not abstract from change. On one reading, the snub serves simply as an example of something that is defined in terms of its matter or some material property.[31] On this interpretation, Aristotle's classification coheres straightforwardly with his views about abstraction in *Met.* M.3. Aristotle is simply distinguishing what is abstracted in mathematical versus natural sciences. The mathematician removes all predicates describing types of matter and change, and this leaves only quantitative features and those relating to a thing's continuity. The natural scientist abstracts various other properties, and maybe even some material or kinetic features, but not all of them. Both study perceptible substances, although not generally as such.[32]

A number of factors should lead us to think that this cannot be the only distinction Aristotle wishes to convey with his contrast between the snub and the concave, however, even if it is one difference the pair is designed to mark out. The example of the 'snub' is excessively obscure and even bizarre if Aristotle is only after an example of something that is not abstracted from matter and defined in terms of change. It is a strain to think of a nose as the matter of the snub, and it is not immediately obvious how change should enter into the definition of the snub. If this reading is correct, Aristotle could have made his point far more clearly by saying simply that natural science studies things like trees. A desideratum of an interpretation is that it explain the work done by the contrast between the snub and the concave specifically, ideally showing this to be an apt example of what Aristotle wants to convey.

A first indication that there is more to the contrast between the snub and the concave is given in *De Anima* III.7, where Aristotle writes that 'one thinks things spoken of in abstraction just as if one were to think of the snub not as the snub, but rather as something separate, as the concave without the flesh in which the concave is,' adding, paradoxically, that '[o]ne thinks mathematical things in this way: though not existing as separate entities, one thinks of them as separate whenever one thinks of them just as they are' (431b12–17, trans. Shields modified[33]). In this passage, the difference between the snub and the concave is not described in terms of the predicates that are abstracted from a nose but in terms of whether the abstracted compound is thought of in connection with the nose. Aristotle's statement implies that thinking of snubness *qua* snubness is not to think about it in separation from its underlying subject, a nose, because if one does think of it as separated from a nose, then one is thinking simply of concavity.

It is only mathematicals that can be thought of as such in isolation from their underlying subjects. Being 'in abstraction' vs 'like the snub' thus concerns not simply what is abstracted, but the thinkability of abstracted things in isolation from the subjects from which they are abstracted.

In what sense does the natural scientist still need to think of the features she studies *as* features of their underlying subject, where the mathematician does not? A start on this question can be made by reflecting further on the example of the snub itself. On the one hand, snubness is the kind of property that essentially inheres in a nose. Nothing other than a nose can be snub, and this is no accident; snubness is as such a feature of noses (*Met.* 1030b30–2). On the other hand, for a nose to be snub is nothing more than for it to be concave. Having taken a nose and made it concave, one has *ipso facto* formed a snub. Thus it seems that we need to carefully balance two ideas: First, that being snub is more than simply being concave, because it is essential to snubness (but not concavity) that it occurs in a nose; second, that a nose's being snub is actually nothing more than its being concave: The snubness of a given nose simply is its concavity.

This delicate relationship cannot be captured solely in terms of which properties are abstracted from a nose as per the procedure in *Post. An.* I.5. For Aristotle is just as interested in the sense in which the snub is the same as the concave as the sense in which it is not. If for a nose to be snub just is for it to be concave, then we cannot say that one obtains the concave by abstracting further properties from the nose, since there is no property that one needs to abstract from a nose's snubness to yield that nose's concavity. The snub and the concave rather represent two possible stances or perspectives that one might take towards a nose considered as concave. If one considers the subject of abstraction to be arbitrary, then one is considering a mathematical property, concavity (*De An.* 431b12–15; cf. *Phys.* 193b33–5). On the other hand, if one keeps in mind that one is considering the concavity *of a nose*, then one obtains an object of natural science. Aristotle's point, in broad terms, is that natural sciences study features specifically *as they occur* in the subjects from which they are abstracted, whereas mathematics studies features *as if they did not occur in any subject* and consequently *as they might occur in any subject at all*.

Aristotle's technical exposition of these ideas comes in the course of the puzzles he raises regarding the snub in *Metaphysics* Z.5, and his solution to those puzzles in the *Sophistical Refutations*.[34] In *Metaphysics* Z, the snub is treated under the banner of 'compounds', or things 'from addition' – notably, as Cleary (1985, 19–20) has observed, a term used to denote the inverse of the operation of 'abstraction' by which Aristotle characterizes the mathematical.[35]

In order to understand Aristotle's technical approach to the snub, it is necessary to lay a little groundwork first. While Aristotle's notion of a *hylemorphic* compound is most well known, this is for Aristotle only one case of a broader idea. In general, a compound for Aristotle is anything which exists as 'one thing in another'. This includes sensible substances, which Aristotle analyses as forms inhering in matter, so that Socrates is a certain form in flesh and bones, but it also includes 'accidental compounds' like the sitting Socrates, which we have already discussed. The snub is a third type of compound, which we can call an 'essential compound'.[36] A nose is, like Socrates in the sitting Socrates, a subject in which something else inheres; but unlike the sitting Socrates, there is an essential connection between the snub and its subject, a nose, since it is noses *as such* which are snub (*Met.* 1030b30–2).[37]

In general, for Aristotle, a compound is a being that exists because one thing is predicated of another, and its existence is contingent on this predication holding. We can therefore define an operator + which, given a term S and a term P that can be predicated of S, forms a new term $S+P$ which denotes the compound of S and P. Accordingly, we can think of abstraction, as a first pass, as the inverse of this operation. If C is a compound of S and P, then $C–S$ ought to retrieve the predicate P.

Aristotle expresses this idea in the *Topics* in the form of an 'abstraction' rule. He says:

> Furthermore, you must note the result of an addition and see whether each added to the same thing fails to produce the same whole; or whether the abstraction of the same thing from each leaves the remainder different. Suppose, for example, someone has stated that a double of a half and a multiple of a half are the same, then, if *of a half* has been abstracted from each, the remainders ought to signify the same thing: but they do not. For 'double' and 'multiple' do not signify the same thing.
>
> Ἔτι ἐκ τῆς προσθέσεως, εἰ τῷ αὐτῷ ἑκάτερον προστιθέμενον μὴ ποιεῖ τὸ ὅλον ταὐτόν. ἢ εἰ τοῦ αὐτοῦ ἀφ᾽ ἑκατέρου ἀφαιρεθέντος τὸ λοιπὸν ἕτερον, οἷον εἰ διπλάσιον ἡμίσεος καὶ πολλαπλάσιον ἡμίσεος ταὐτὸν ἔφησεν εἶναι. ἀφαιρεθέντος γὰρ ἀφ᾽ ἑκατέρου τοῦ ἡμίσεος τὰ λοιπὰ ταὐτὸν ἔδει δηλοῦν· οὐ δηλοῖ δέ· τὸ γὰρ διπλάσιον καὶ πολλαπλάσιον οὐ ταὐτὸν δηλοῖ. (*Top.* H.1, 152b10–15, trans. Lewis modified)

As Frank Lewis (2013: 114) has noted, this rule is modelled on the famous axiom that equals taken from equals leave equals, carried over to relations of sameness among compounds.[38] If it were the case that the double of a half and a multiple of a half were the same, we should be able to subtract 'of a half' from both sides of the equation, as it were, but that would yield the absurd result that 'double' and 'multiple' are themselves the same. Aristotle's emphasis in this passage is on how the rule may be used to reject such specious claims of identity, but clearly it also permits us to infer identities between properties based on the identity between compounds. Given that being double of a number is the same as being a multiple of two of a number, we may rightly infer that being double is the same as being a multiple of two. In general:

> (Abstraction Rule) If $C=C'$, then for any S such that S is C and S is C', $C–S=C'–S$[39]

This rule, then, seems both to be plausible and indeed to follow pretty immediately from the ideas of abstraction and compounding presented so far. Yet it will be noticed that in certain cases the rule causes trouble, and the snub is a case in point. As Aristotle says, 'if a snub nose is the same as a concave nose, then snubness will be the same as concavity'.[40] This inference is an application of our principle, as Lewis (2013: 100) observes: From the identity of the compounds, snub nose and concave nose, we infer the identity of the properties in those compounds, namely snubness and concavity, by abstracting the nose from both sides.

This conclusion is already problematic, since there are features of snubness which are not features of concavity in general. Aristotle does not specify what these are, but one which cannot fail to occur to Aristotle's audience is the famously unsightly character of snubness exemplified in Socrates' legendary nose (*Theaet.* 143e–144b), and Aristotle discusses snubness as a deviation from the 'beautiful' ideal of straight-nosedness in *Politics* V.9 (1309b23–29; cf. *Rep.* 474d–e). The problem is that snubness is ugly (by Athenian tastes, at least), yet there is nothing intrinsically ugly about concavity. The concavity of the upper leg on a statue of Dionysus has the opposite aesthetic quality. Concavity is simply a mathematical property, and if it has any intrinsic aesthetic status at all, it is beautiful for the precision and order that it represents (cf. *Met.* 1078a36–b2). So we have our first paradox:

1. snub nose = concave nose (premise)
2. being snub = being concave (abstraction rule)
3. being snub is intrinsically ugly
4. being concave is not intrinsically ugly

Further trouble is caused when we apply the result of this abstraction to specific cases. If being snub is the same as being concave, then by symmetry being concave is the same as being snub, but this then yields the result that anything which is concave is snub. This is obviously wrong: The inner edge of a semicircle in a Euclidean diagram is not snub, nor is the curve of Dionysus's thigh. Being concave coincides with being snub only when the concavity in question occurs in a nose.

In *Metaphysics* Z.5, Aristotle considers two ways of avoiding this paradox, both of which he takes to fail.[41] First, we might reject the claim that a snub *nose* is a concave nose and instead say that *snub* (period) is defined as a concave nose. That will not work, however, for we can still speak of a snub nose, but then by substituting 'snub' for its definition, a snub nose will be a concave nose nose, which is nonsense (*Met.* 1030b30–5). The remaining possibility is that we forego reference to concavity altogether and simply define the snub as a snub nose; but in this case we get an even worse result, namely an infinite regress, since we can then substitute 'snub nose' for 'snub' in the definiens to yield the statement that a snub is a snub nose nose, and again that it is a snub nose nose nose, etc. (1030b35–1031a1).

These three alternatives (snub nose $=_{\text{def}}$ concave nose, snub $=_{\text{def}}$ concave nose, snub $=_{\text{def}}$ snub nose) come to three ways of spelling out the idea that the snub must be defined with reference to a nose, and Aristotle raises puzzles for all of them. Z.5 is thus aporetic as concerns the definition of snubness, leaving us only with the conclusion that if there is a definition of things like the snub, it will be defined 'in a different way' than a substance is (1031a7–10). But in fact the situation is worse than this. For even if we deny that 'snub nose = concave nose' is a *definition* of the snub, we are still left with the problem resulting from the use of the abstraction rule, namely that this equation appears to yield the unwelcome result that snubness is the same as concavity in general.

Aristotle's way of avoiding this result, and thus resolving the puzzle, comes in *Sophistical Refutations* 31,[42] where he introduces the idea that certain properties may mean (δηλοῖ, 185b38) something different in the context of a compound and in abstraction. In particular, 'concave' in 'concave nose' does not have the same signification as it does in 'concave leg' (181b39–182a1): In the one case it refers to snubness, while in the other it refers to bandiness.[43]

The nose, then, leaves an 'imprint', as it were, in the terms after abstraction. Taking equals from equals consequently does not, at least in this case, allow us to infer a completely general identity between properties but only an identity of those properties *as they apply to that subject*. What this amounts to is a restriction of the abstraction rule to a form that requires that we index the relevant property to its original subject:

(Innocent Abstraction) $(S + P) - S = P_S$

That is, we index the predicate to the subject of the compound from which the predicate is abstracted. In the case of the snub, which Aristotle treats as a nose compounded with concavity, this reduces to the principle that snubness is, as Aristotle says, concavity *of a nose* (ῥινός, 182a4). This avoids the paradox, because the equation 'snubness = concavity of a nose' does not permit any problematic substitution into 'snub nose'. One can only infer that a snub nose is the same as 'a nose possessing the concavity of noses', which is 'not at all absurd' (182a5–6).

Nevertheless, Aristotle does not simply dispense with the stronger principle, and this is where the foregoing discussion can shed light on Aristotle's philosophy of mathematics. For Aristotle thinks that mathematical sciences are *not* of things 'like the snub' but rather things 'in abstraction'. Now, if being 'like the snub' means that abstraction is always incomplete (in the sense that the abstracted property must still be understood as the property as it applies to its original subject), then things 'in', 'from' or 'through' abstraction must differ in that they allow complete abstraction from their subjects. Accordingly, we can say that for Aristotle mathematical objects are abstract in the sense that the unrestricted abstraction principle holds. That is,

If M is a mathematical object that is abstracted from S, then $(S + M) - S = M$

This shows clearly that the difference between mathematical objects and objects of natural sciences, for Aristotle, is not merely a matter of what is abstracted in each case. The difference is in the relation the abstracted thing bears to the subject from which it is abstracted. Mathematical objects are distinctive in that the subject of a mathematical compound can be removed from it without residue.[44]

Once we see this, we can see why Aristotle takes mathematical objects to differ from the objects of natural science in that they may be safely treated as if they were substances. The problem with treating properties which are not substances as substances in the natural sciences is that one then treats them in isolation from the subjects in which they always inhere. This does not render sciences which take non-substantial beings as their subjects generally illicit: Aristotle himself engages in scientific theorizing about colours and sounds,[45] and his works are replete with discussions of theories about natural processes, from the formation of rainbows to the reproduction of fish. It also doesn't mean that a limited kind of abstraction or reification is out of place in these sciences. A scientist may still talk about colours without every time mentioning that they are colours of a surface. What it does mean is that the results of that science remain implicitly indexed to some relevant subject. If I think I have defined snubness, what I have really done is defined what it is for a *nose* to be concave. Trouble ensues if I forget this and think that I have defined the abstract quality of snubness, which is simply concavity.

In general, we can say that in natural sciences, propositions apparently concerning abstract qualities, like being blooded or thundering, turn out to really be masked statements concerning the specific mode in which these properties inhabit their subjects. To use Aristotle's toy example in the *Posterior Analytics*, for thunder to be the extinguishing of fire really means that a *cloud's* thundering is the same as a *cloud's* particular way or mode of extinguishing fire. Whereas the restriction of thundering to clouds is trivial (since nothing thunders except for a cloud), the restriction of extinguishing fire to clouds is not. Aristotle does not mean that thunder is identical with the extinguishing of fire in the abstract, which would commit him to the implausible view that snuffing out a candle is a very small thundering. Instead, he means that thunder is the specific way that extinguishing fire exists or takes place in clouds. It can still be true to say, as a numinological proposition, that thunder is the extinguishing of fire, but the significance of this being a numinological proposition is that all of the properties mentioned need to be understood as properties of, or processes within, clouds in particular.[46]

Nothing analogous holds in the mathematical case. Even though, ontologically speaking, mathematical objects exist as compounds whose subjects are physical, sensible bodies, they can be defined in abstraction from these subjects. This is because the mathematical properties are precisely those whose essence is in no way determined by the subjects they inhere in. To be concave, in other words, is not to be understood as the form that some more abstract property takes when it exists in a particular kind of thing, whereas other properties (like snubness) may be understood as the manifestation of mathematical properties in a specific sort of subject. And so, while the original subjects *can* be reintroduced, viz. an inscription's being triangular is its having three sides, the specification will *always* be trivial in the sense that the relevant subject is perceptible body quite generally. There is no other kind of subject to which these properties could apply.

A useful way to think about this is as a distinction between subjects which are 'inert' versus 'reactive' in the formation of a compound. When a compound is formed by attributing triangularity to a body or a diagram, the features that triangularity imparts are simply those of triangularity as such.[47] By contrast, Aristotle thinks it characteristic of natural sciences that they study features which result from the inherence of predicates in a certain kind of subject but which are not features that the predicate would impart to just any subject. This is neatly captured by the case of the snub: The allegedly negative aesthetic features of a snub nose are neither inherent to concavity nor to noses as such. They are features that arise only from the combination of concavity with the nose. This is also why a natural science must not attempt to abstract in such a way that the subject of the things studied is permanently forgotten, for if it does so, by moving from the consideration of e.g. life in animals to life in the abstract, it will miss the bread and butter of its inquiry, which is a study of how the soul manifests in different kinds of living things (*De An.* 414b25–28).

This has important consequences for the projects of scientific explanation in mathematical and natural sciences respectively. Since a natural scientist studies reified properties as they occur in subjects, it will be open to the natural scientist, and indeed necessary, to explain some of their features by appealing to their inherence in these subjects.[48] Its being in a cloud, for instance, is indispensable to explaining the characteristic noise of thunder. This noise cannot be explained simply on the basis of a cloud's existence or the extinguishing of fire *in abstracto*. The same presumably holds for Aristotle's other two recurrent examples in *Post. An.* II, leaf-shedding and the eclipse. Perhaps not all leaf-shedding occurs as the coagulation of sap, but only in broad-leafed plants, and the *solar* eclipse, of course, does not occur by obstruction of the sun's light due to interposition of the earth (cf. 98a38–b4). Even where the relevant subjects are not mentioned, then, they are always in the background, implicitly determining the way the explananda and explanans are to be interpreted.

Not so for mathematical properties. While actually existing as features of physical, perceptible things, it would be out of place for a mathematician to appeal, even implicitly, to the nature of this substrate in explaining why a mathematical theorem holds. Mathematics as a discipline, then, is a study of abstractions in that its objects permit abstract relations between their properties to be studied in a way that natural sciences do not. We can therefore draw a distinction between being a result of abstraction in a broad sense and being a 'thing from abstraction' in Aristotle's narrow technical sense. Many sciences, as Aristotle tells us in M.3, consider subjects *qua* this or that, and in this sense many sciences study abstractions, but only in mathematics does this result in the predicable being's status as predicable itself being abstracted away. The function of the contrast with the snub is not simply to emphasize that mathematics abstracts from certain features like motion that are essential to natural sciences,[49] but to illustrate this structural difference.

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Notes

1. Conspicuous examples occur at I.1, 71a19–24, where 'triangle' (τρίγωνον) is classified as a 'particular' that is 'not said of any underlying subject' (cf. I.13, 79a8, where this is extended to all mathematical objects); at I.4, 73a35–37, where 'line' and 'triangle' occur as subjects of predication *per se*; at I.4, 73b30–31, where 'having angles adding to two right angles' is said to hold of triangle *qua* triangle (cf. 85b6); at I.5, 74b2–3, where triangle is taken to be the primary subject of a universal predication; and at I.10, 76a34–36, where the existence of the monad is listed as something a science takes for granted (cf. 93b24–25). See Goldin 1996: chs. 2–3 on the broad sense of 'substance' in which Aristotle treats non-substantial entities as substances in the *Post. An*.
2. It is also raised in *Met.* B.5. On the *aporiai* and their relation, see Katz 2018.
3. Mueller 1970: 157; Katz 2022: 156, 162; Hussey 1991: 106; Corkum 2012: 1072; Cleary 1995: 446. Pfeiffer 2018: 32 is more circumspect on this point.
4. The term 'kooky object' is coined by Matthews 1982. See further Cohen 1996; 2008.
5. Extending this to the case of lower-dimensional geometrical objects requires some technical subtlety. The basic idea is that points (zero-dimensional objects) exist as the limits and divisions of line segments (one-dimensional objects), which in turn exist as the divisions of two-dimensional objects, which themselves bound or limit three-dimensional bodies. See Katz 2022: 170–73 for details.
6. Katz 2019: 468; cf. Katz 2022: 180.
7. Annas translation modified. All translations from *Met.* M are from Annas 2003 unless noted.
8. See Pfeiffer 2018: 27–28n.7, 32 for a similar distinction.
9. Cf. Katz 2022: 147.
10. On this point further, see Mendelsohn 2023.
11. Cf. Hussey 1991: 109.
12. Scholars agree that Aristotle takes most of the triangles, circles, etc. encountered in sensory experience to be imprecise specimens of these geometrical kinds. There is debate on whether Aristotle's theory requires at least *some* natural or

artificial sensible shapes to be precise geometrical objects. *Pro* see Lear 1982: 176; Katz 2019: 476; Katz 2022: 163; *contra* see Mueller 1970: 157–58; Annas 2003: 29 and Modrak 2001: 120. I will remain neutral on this issue in this paper. Aristotle's distinction between what a science is of and the extension of things it studies opens up at least logical space for precise mathematical objects without precise sensible particulars, but it is another question, which I will not tackle, whether Aristotle can maintain the existence of precise mathematical objects without there being at least some precise sensible particulars.

13. See especially Matthews 1982; Cohen 1996, 2008.
14. See further Katz 2022: 154–58.
15. For details, see Katz 2021, 2023.
16. Cf. Cohen 2008: 15.
17. See especially 1087a10–25.
18. See Katz 2022: 164n.69; Katz 2019: 495–97.
19. Contra Katz 2022: 164n.69. This is what Mendell 2019: sec. 6 calls the 'problem of plurality'.
20. Pettigrew 2009 argues that, for Aristotle, mathematicians study actual or potential parts of bodies. Katz (2022: 173–74, 183–89) rejects this reading in favor of the view that precisely constructed bodies already actually have precise mathematical properties. Note that even if Aristotle does hold that sensible bodies have mathematically precise features (an issue I will remain neutral on in this paper), this deals only with one of the pressures to distance mathematical objects from sensible things, namely the problem that perceptibles are allegedly imprecise. It does not deal with the problem regarding perishability. Katz also objects that treating mathematical objects as parts of bodies would misconstrue their ontology, making them into substances. This is a fair criticism of Pettigrew's view, but it does not apply to my proposal here, which is that the properties entering into mathematical compounds may include potentialities. A proponent of my interpretation also needn't follow Pettigrew (2009: 248) in holding that the identification of geometrical objects is what actualizes them, a position Katz also rightfully wishes to avoid.
21. Mueller (1970: 158) argues that *Metaphysics* M.3 does not adequately express Aristotle's ultimate view. I take this to be an interpretive measure of last resort. Like most commentators, I will assume that Aristotle has a consistent view about mathematics since I do not think the texts are impossible to render consistent or require a great stretch to do so.
22. Katz 2022: 153, 167n.75; Katz 2022: 476; Cleary 1985: 21; Pfeiffer 2018: 30–31.
23. Here '2R' refers to the property of having internal angles whose sum is equal to the sum of two right angles. On this property in Aristotle see Tiles 1983.
24. On this chapter, see Hasper 2006.
25. Cf. Pfeiffer 2018: 35.
26. Cf. Katz 2022: 152, 162–63.
27. For a discussion of this passage in relation to Aristotle's philosophy of mathematics, see Cleary 1995: 491–93; Cleary 1985: 14–18. I agree with Cleary that abstraction should not be taken to denote a third mode of learning besides induction and deduction.
28. ᾗ δὲ μὴ τοιούτου σώματος πάθη καὶ ἐξ ἀφαιρέσεως, ὁ μαθηματικός (*De An.* 403b14–15); πάλιν δ᾽ ἐπὶ τῶν ἐν ἀφαιρέσει (429b18); τὰ δὲ ἐν ἀφαιρέσει λεγόμενα (431b12–13). See also *Nic. Eth.* 1142a11–19 with Cleary 1995: 491 and *Phys.* 193b22–a12.
29. Aristotle also holds that the mathematician studies her objects *qua* changeless, but he is less clear about whether mathematical objects are in reality changeable (cf. *Met.* 1026a7–10, 14–15).

30. One might wonder whether there is after all a parallel in the natural sciences, since all sciences, including natural sciences, study many individuals in so far as they all have one and the same form (cf. *Post. An.* 74a30–32, 77a9–12). Should we say that natural sciences like biology, therefore, study many things *qua* one, and hence that they also attribute to their subjects properties contrary to those they really have? No, for there is no true parallel here, as a specification of the relevant subjects shows. The mathematician treats a *given* mathematical triangle – the object contemplated in a geometrical proof, say – as existing separately from its matter in the geometrical diagram, even though, ontologically speaking, it does not exist separately from the diagram. For there to be a parallel in the biological case, a biologist would need to study an individual within her domain of study – an organism, say – under some aspect contrary to its actual nature. She does not, however: Any individual she studies really is one, it is not a multiplicity studied *qua* one.
31. See Pfeiffer 2018: 37–42 for an interpretation along these lines.
32. Cf. Katz 2019: 490–92.
33. τὰ δὲ ἐν ἀφαιρέσει λεγόμενα ὥσπερ, εἴ τὸ σιμὸν ᾗ μὲν σιμὸν οὔ, κεχωρισμένως δὲ ᾗ κοῖλον εἴ τις ἐνόει, ἄνευ τῆς σαρκὸς ἂν ἐνόει ἐν ᾗ τὸ κοῖλον – οὕτω τὰ μαθηματικά, οὐ κεχωρισμένα , ὡς κεχωρισμένα νοεῖ, ὅταν νοῇ ἐκεῖνα. Following the manuscript reading against Ross's addition of ἐνεργείᾳ (after Bywater) at line 14, as Shields evidently also does. Shields adds 'other' after 'one thinks' at the beginning of the translated passage, apparently taking τὰ […] εἴδη at 431b2 to be synonymous here with τὰ […] ἐν ἀφαιρέσει at 431b12 so that τὰ δὲ ἐν ἀφαιρέσει needs to refer to a sub-class of the things in abstraction. I find the text clearer if we take Aristotle to be treating τὰ εἴδη and τὰ ἐν ἀφαιρέσει as separate classes of intelligible entities in turn. In any case, Aristotle makes explicit that the abstractions he is discussing include objects of mathematics at 431b15.
34. Pfeiffer 2018 also proposes to elucidate Aristotle's view of mathematical science (although not mathematical objects) by comparison of the concave with the snub. He explicitly brackets *Met.* Z.5 in his exposition, since he thinks that it is used there only as an example for something that lacks a definition (Pfeiffer 2018: 38). In my view, this is a serious mistake. Aristotle is not definitively claiming that the snub cannot be defined (see 1031a8); he is raising puzzles whose resolution is supposed to reveal the distinctive way that things like the snub need to be defined.
35. See *De Cael.* 299a13–17.
36. Often, the snub is treated as a hylemorphic compound in the literature (e.g. Balme 1984: 1). This is understandable given that Aristotle frequently employs it as a *model* for hylemorphic compounds, but for our purposes more precision is required, since it is the ontology of the snub itself which is of interest rather than its application in elucidating hylemorphic metaphysics. The nose is not literally the matter of the snub, it is the subject (or 'substratum', Uhlmann 2017: 6) in which snubness inheres. Aristotle often notes and works with the analogy between matter and underlying subjects: On this further, see Code 2015.
37. Plausibly, the sense in which the connection is essential is the second sense of καθ' ἁυτό described at *Post. An.* 73a37–73b1.
38. The resemblance is also noted in Cleary 1985: 19, but Cleary explicitly abstains from making use of this analogy to explain Aristotle's notion of mathematical abstraction.
39. Here I write '=' for the relation of sameness rather than equality. The symbol should be taken this way wherever it is employed in this paper.
40. *Metaphysics* Z.5, 1030b28–30.

41. My presentation here closely follows Lewis 2013: ch. 4.
42. For a more pessimistic appraisal of Aristotle's proposed solution in the *Sophistical Refutations*, see Balme 1984: 3. Balme appears to take *Sophistical Refutations* 31 to be attempting to explain how 'snub' might be defined without reference to a nose (see esp. p. 6), but this is not Aristotle's task there. Aristotle is trying to show how he can avoid paradoxes of 'babbling' while also avoiding the paradoxical claim that snubness is identical to concavity in general.
43. Balme (1984: 3) takes Aristotle to propose two distinct, incompatible ways of resolving the paradox in *Soph. El.* 31, one in which 'the expression concave nose is allowed on the grounds that it gives a special sense to concave' (181b38–182a2) and one in which the expression is 'disallowed'. But Aristotle is not sanctioning the expression 'concave nose' at 182a3–6. He is denying that the expression 'snub', where this refers to the *quality* (πάθος, 182a4) of snubness, can be equated with 'concave nose' (as opposed to concavity *of* a nose). This is compatible with his claim at 181b38–182a2 that the meaning of 'concave' differs according to context.
44. Note the significance of the qualification that S is the subject *from which M is abstracted*. On the view being argued for, the contrast marked by 'things from abstraction' vs 'the snub' is one between the relationship that an object of natural science bears to the object from which it is abstracted and the relationship a mathematical object bears to the subject from which it is abstracted. In both cases, Aristotle holds, the subject is typically a natural body (*Phys.* 193b23–25; presumably looking at an inscription in a diagram counts as observing a natural body). The point is not about the relationship that mathematical properties bear to mathematical objects, since mathematical objects are already abstractions. On this further see note 47.
45. See Sorabji 1972 and Johnstone 2013 on Aristotle's theories of colour and sound respectively.
46. See 93a22–3, 93b7–13 and 94a3–9 with Barnes 1981. The major premise of the syllogism Aristotle describes at 93b7–13, 'thunder is the extinguishing of fire', is presumably a definition in the sense of an 'indemonstrable account of what something is' (94a11–12). The full definition of thunder, corresponding to 'a demonstration of what something is, differing in arrangement' (94a2), makes explicit that the extinguishing of fire in question is the sort to take place in clouds.
47. To be clear, the point here is about the difference in the result when a mathematical object forms a compound with its true ontological subject (typically a natural body) versus the result when an object of natural science does. It is *not* a point about the inertness of mathematical properties: After all, concavity is a mathematical property, and when attributed to a nose, it produces something new, snubness. Nor is it a point about the inherence of one mathematical object or property in another, as when one says that triangles and squares both have the property of being figures. On the view being argued for, these are not truly predications of properties of substances, since triangles are in turn compounds of properties and natural bodies, and so, ontologically speaking, the square and triangle are not the true subjects of these predications. They can be treated for the purposes of scientific inquiry as if they were freestanding subjects precisely because of the status of the triangle as a special kind of compound, but that does not mean that the properties of triangles function semantically like the concave does in relation to its real subject, a body. Hence it is open to Aristotle to maintain that 'odd' might bear a relationship to number analogous to the relationship the snub bears to a nose (*Soph. El.* 173b8–9;

cf. *De An.* 429b18–19), or that 'figure' has a different meaning when attributed to triangles and squares, as he does in *De An.* 414b20–6 on the interpretation of Ward 1996. Thanks to Rory Hanlon for discussion on this point.

48. Cf. Charles 2008: 6.
49. Contra Pfeiffer 2018: 36.

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